Moral hazard and the quest for linear contracts*

Marcus Opp†

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Abstract

This paper establishes necessary and sufficient conditions on utility functions and output distributions for (piece-wise) linear contracts to arise in the standard static principal-agent framework of Holmström (1979), rationalizing observed contracts involving stocks or call options. I present a complete characterization of optimal contracts for a key case: exponentially distributed output and log utility for the agent. When the principal seeks to implement effort levels below a threshold, the optimal contract is affine, consisting of a fixed wage and a constant output share. Effort levels above this threshold are implementable, but no optimal contract exists without an additional minimum-wage constraint. If such a constraint is introduced, an optimal contract exists and is piecewise linear, consisting of a fixed wage and a call option on output. I derive a closed-form solution for infimum compensation cost in the unconstrained problem (without a minimum wage).

Keywords: Linear Contracts, Contract Theory.

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1 Introduction

The classical static principal agent model of Holmström (1979) asserts that the likelihood ratio of output realizations is a sufficient statistic for designing optimal compensation contracts. Yet, as successful the informativeness principle was and is for our general understanding of optimally addressing moral hazard, the model has been criticized because its predictions do not readily align with the simple, linear contracts often observed in practice. In response, subsequent research either assumes linear contracts even though they are not optimal (e.g., Holmström and Milgrom (1991), Bolton and Dewatripont

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†Stockholm School of Economics, CEPR, ECGI and FTG; e-mail: marcus.opp@hhs.se

(2004) or departs from the classical setup (e.g., Holmström and Milgrom (1987), Innes (1990), Hébert (2017), Yang (2019), Mattsson and Weibull (2023)) to obtain optimal, linear contracts.

This paper demonstrates that one need not abandon the Holmström (1979) framework to obtain linear contracts as the optimal resolution of the trade-off between incentives and insurance. I establish necessary and sufficient conditions on the agent's utility function and the output distribution for affine (or piecewise linear) contracts to arise and provide conditions for existence. In particular, affine contracts emerge generically when (a) the agent exhibits logarithmic utility and (b) output is drawn from a Gamma distribution (with the exponential distribution as a special case). If the principal aims to induce agent effort below a threshold, an optimal contract exists and can be implemented by paying the agent a base wage along with a constant share of output. Effort levels above this threshold are implementable, but an optimal compensation contract fails to exist in the absence of an additional lower bound on pay (e.g., a minimum wage), a phenomenon analogous to the classical, log-normal example by Mirrlees (1999). If such a constraint is imposed, existence is ensured and the optimal contract becomes piecewise linear, consisting of a base wage and a call option on output with a strike price that decreases with the minimum wage. Taking the limit as the minimum wage goes to zero yields a closed-form solution for the infimum compensation cost of the unconstrained problem (without a lower bound on pay).

I derive these results within the standard setup of Holmström (1979) featuring a risk-neutral principal and a risk-averse agent with additively separable preferences in consumption and effort. The agent's unobservable effort, denoted by a, generates stochastic output x. Under regularity conditions, Holmström (1979) shows that the optimal compensation schedule $c(x)$ satisfies

$$\frac{1}{u'(c)} = \lambda_{PC} + \lambda_{IC}\, L(x|\, a)\,, \tag{1}$$

where u is the agent's utility from consumption, $L(x|\, a)$ is the likelihood ratio associated with output x given effort a, and $\lambda_{PC} > 0$ and $\lambda_{IC} > 0$ are the Lagrange multipliers corresponding to the participation and incentive constraints, respectively. The quest for linearity is to establish conditions so that (1) generates affine contracts regardless of the agent outside option or effort cost function, both of which affect the Lagrange multipliers. It is now immediate that affine contracts emerge independently of the value of the Lagrange multipliers if and only if both the inverse marginal utility and the likelihood ratio are themselves affine functions. Logarithmic utility ensures the former condition, while the Gamma distribution satisfies the latter.

Affine contracts are fully characterized by two parameters: a base wage and an output share. Under such a contract, the agent's expected utility is a strictly concave function of effort, ensuring that the first-order approach holds for any convex effort cost function.[1] Constant relative risk aversion implies that effort incentives of a given output share are lowered as the base wage increases. In particular, with logarithmic utility, a sufficient statistic for effort incentives is the *ratio* of the output share to the base wage. Intuitively, higher effort requires a higher ratio, i.e., a larger variable component relative to fixed pay. Maximal incentives under an affine contract are achieved when the principal relies solely on variable pay (and no fixed wage). Yet, even this pure linear scheme fails to provide sufficient incentives when the principal aims to induce sufficiently high effort.

For sufficiently high-effort, the issue is not implementability per se, but implementability via an affine contract. For any exogenously given wage level, the principal can induce high effort levels by making the incentive scheme more convex, e.g., by awarding an output share only if output exceeds a cutoff, effectively offering a call option on output. This modified contract is exactly optimal if the principal faces an additional minimum wage constraint. The option strike price is optimally larger as the minimum wage increases as to counteract the less severe punishment conditional on falling short of the strike price. I provide a closed-form solution for compensation costs as the minimum wage tends to zero. These costs represent the infimum compensation costs in the unconstrained problem (without the minimum wage). Since the call option strike price converges to zero as the minimum wage converges to zero, the resulting limiting contract is akin to a linear scheme from the risk-neutral principal's perspective, yet the prospect of an infinitely severe (zero-pay) punishment, even if realized with zero probability, provides strong incentive effects on the risk-averse agent.

I conclude the paper with the equilibrium action choice problem of the principal, step 2 of the Grossman and Hart (1983) procedure. I show that intermediate values of the agent's outside option both ensure that the principal implements actions below the existence cutoff, yielding affine contracts, and obtains positive net profits.

Literature My paper relates to two classical strands of the contract theory literature. First, it builds on the literature on optimal linear compensation contracts, originating from the seminal work of Holmström and Milgrom (1987). Unlike their framework, in which linearity emerges due to repeated moral hazard and independent compensation

[1] The first-order approach holds even though the exponential distribution only satisfies the monotone likelihood ratio property, but not convexity, so the validity does not follow from the sufficient conditions established by Rogerson (1985).

schemes over *time* (due to constant absolute risk aversion), this paper shows that linearity can arise across *states* due to the properties of the likelihood ratio (with constant relative risk aversion).[2] To the best of my knowledge, this is the first paper to demonstrate that linearity can emerge within the classic Holmström (1979) framework. Hence, abandoning this foundational model—widely used in graduate studies—may not be necessary to obtain realistic and tractable contracts.

Second, this paper relates to the literature on the existence of optimal compensation contracts. Mirrlees (1999) demonstrated that when utility functions are unbounded (such as for log utility) and the output distribution is log-normal, optimal contracts do not exist. Instead, first-best compensation costs can be arbitrarily approximated by imposing severe penalties on the agent for exceptionally low output. Müller (1999) extends this insight by showing that the problem arises for all distribution functions for which the likelihood ratio is unbounded below, allowing the principal to perfectly detect deviations. However, the exponential distribution considered in this paper features a likelihood ratio that is bounded below. Hence, most closest related is the paper by Moroni and Swinkels (2014) who show that bounded likelihood ratios are not sufficient to ensure existence of an optimal contract. In particular, they show that if the agent's utility function diverges at a finite consumption level, such as for log utility, the existence problem is severe in that existence depends on the cost function (and the implemented effort level).[3] This result explains why an optimal affine contract only exists in my setting if the implemented effort level is sufficiently low.

2 Model

I consider the standard, static principal-agent model à la Holmström (1979). The principal observes output signals X according to the density function $f(x|a)$, which is parameterized by the agent's action $a \in \mathbb{R}^+$. To implement an action a, the principal designs a compensation scheme c as a function of realized output x. The principal is risk-neutral and the agent has an additively separable utility function in consumption c and effort a, $U(c,a) = u(c) - k(a)$ where $u(c)$ is strictly concave and her cost of effort $k(a)$ is strictly increasing and convex.

Let $\mathbb{E}^a$ denote the expectations operator given action a, then the minimum-cost compensation contract to implement action a solves the following program:

[2] In contrast to Innes (1990) and Hébert (2017), my paper considers a risk-averse agent.

[3] Kadan, Reny and Swinkels (2017) build on this insight and provide sufficient conditions for existence, one which is a bound on the penalties that can be imposed on the agent.

Problem 1

$$W(a) := \min_{c(x)} \mathbb{E}^{a}[c(x)] \qquad s.t.$$

$$\mathbb{E}^{a}[u(c(x))] \geq k(a) + \underline{u}. \tag{PC}$$

$$a = \arg\max_{\tilde{a}} \mathbb{E}^{\tilde{a}}[u(c(x))] - k(\tilde{a}). \tag{IC}$$

It is customary in the literature, see Bolton and Dewatripont (2004), to assume that the agent's optimization problem in (IC) is characterized by a first-order condition.[4] Assuming that this first-order approach holds, (IC) can be written as:

$$\mathbb{E}^{a}[L(x|a)\, u(c(x))] = k'(a), \tag{IC-FOC}$$

where $L(x|a) := \frac{\frac{\partial f(x|a)}{\partial a}}{f(x|a)}$ denotes the continuous-action likelihood ratio. Pointwise optimization, see Holmström (1979), implies that the optimal compensation contract satisfies the fundamental optimality condition (1) highlighted in the introduction, and $\lambda_{PC} > 0$ & $\lambda_{IC} > 0$ are the respective Lagrange multipliers on (PC) and (IC-FOC). Economically, (1) implies that the marginal cost of transferring utility to the agent is an affine function of the likelihood ratio. As is well-known, optimality condition (1) typically generates non-linear contracts in terms of realized output. In fact, unless the monotone likelihood ratio property (MLRP), see Milgrom (1981), is satisfied, the optimal compensation scheme c might not even be increasing in output x.

3 Affine contracts as optimal contracts

Since linearity in output is a prevalent feature of compensation schemes observed in the real world (and assumed in many applied theory papers), it is thus of special interest to determine the class of utility and probability distribution functions that generate such contracts.

Lemma 1 *The solution to (1) generates an affine (and increasing) compensation scheme for all values of the agent outside option $\underline{u}$ and cost functions $k(a)$, if and only if the following two conditions are satisfied.*

1. *The agent has generalized log utility $u(c) = \ln(c - \underline{c})$ for some constant $\underline{c}$, and*

[4] Rogerson (1985) and Jewitt (1988) provide conditions for the validity of this approach.

2. *the probability density function, $f(x|\, a)$, can be factorized as follows:*

$$f(x|\, a) = \kappa(a)\, z(x)\, e^{-\delta(a)x}, \tag{2}$$

for positive-valued functions $\kappa(a)$ and $z(x)$. $\delta'(a) < 0$ ensures that the compensation scheme is strictly increasing in x.

These two conditions are sufficient as they imply that the inverse marginal utility, $\frac{1}{u'(c)}$ is affine in consumption, and the likelihood ratio is affine in output as $L(x|\, a) = \frac{\kappa'(a)}{\kappa(a)} - \delta'(a)\, x$ while $\delta'(a) < 0$ ensures the monotone likelihood ratio property. These conditions are also necessary to guarantee that affine contracts emerge independently of the values of the Lagrange multipliers λ_{PK} and λ_{IC} which encode the agent outside $\underline{u}$ and effort cost function $k(a)$.[5]

Corollary 1 *The Gamma-distribution family satisfies Condition (2) with $\kappa(a) = \frac{a^{-\eta}}{\Gamma(\eta)}$, $z(x) = x^{\eta-1}$ and $\delta(a) = \frac{1}{a}$ for $x, \eta > 0$.*

Within this family, the exponential distribution, which is obtained by setting $\eta = 1$, is the most prominent example in economics. In this special case, the agent action a determines mean output and positive likelihood ratios are obtained whenever realized output exceeds the mean, i.e., $L(x|\, a) = \frac{x-a}{a^2}$, so that $\mathbb{E}^a[L(x|\, a)] = 0$ and $\mathbb{V}ar^a[L(x|\, a)] = \frac{1}{a^2}$. We will now fully characterize the solution to Problem 1 for the exponential distribution and log utility, i.e., $\underline{c} = 0$.

For any desired action $a > 0$, it is instructive (and without loss of generality) to specify an affine compensation contract as follows

$$c(x) = w + w\beta \frac{x}{a}, \tag{3}$$

where $\frac{x}{a}$ is scaled output (normalized by mean output a), $w \geq$ refers to the fixed wage and $\beta \geq 0$ is the *ratio* of the agent's scaled output share $w\beta$ to the fixed wage w. With log utility, this ratio β is a *sufficient statistic* for incentives provided by affine contracts. This follows from the fact that (IC-FOC) under log utility and an affine compensation scheme (3) satisfies:

$$\mathbb{E}^a\left[L(x|\, a) \ln\left(w + w\beta \frac{x}{a}\right)\right] = \mathbb{E}^a\left[L(x|\, a) \ln\left(1 + \beta \frac{x}{a}\right)\right], \tag{4}$$

[5] The qualifying statement "independently of the values of the Lagrange multipliers" rules out knife-edge cases in which affine contracts emerge only for a particular combination of the agent's outside option and effort cost function.

which uses the fact that the likelihood ratio is zero in expectation, $\mathbb{E}^a [L(x|a)] = 0$. When the parameters β and w implement action a, the principal's expected wage costs then satisfy

$$\mathbb{E}^a [c(x)] = w(1+\beta), \tag{5}$$

so that β also measures the expected percentage bonus on top of the base wage w.

Proposition 1 (Exponential Distribution) *A solution to Problem 1 exists if and only if $a \leq \bar{a}$, where the threshold $\bar{a} > 0$ solves $ak'(a) = 1$.*
Let $\mathrm{Ei}(y) := -\int_{-y}^{\infty} e^{-t} dt$ denote the Exponential Integral and $G(y) := 1 + \frac{1}{y} e^{\frac{1}{y}} \mathrm{Ei}\left(-\frac{1}{y}\right)$, then the optimal contract parameters β^* and w^*, see (3), to implement action a satisfy:

$$G(\beta^*) = ak'(a), \tag{IC*}$$

$$\ln w^* = \underline{u} + k(a) - \beta^* (1 - ak'(a)). \tag{PC*}$$

Given this contract, the first-order approach is valid for any convex cost function $k(a)$.

In sum, for $a \leq \bar{a}$ we obtain a compensation contract that is affine in output x. The optimality conditions for the contract parameters β^* and w^* are intuitive. The right hand side of (IC*), $ak'(a)$, measures the strength of *required* incentives and is, hence, intuitively increasing in the implemented action a (formally due to convexity of k). The required incentives need to match the *provided* incentives by the contract on the left hand side of (IC*), which are captured by the strictly increasing function $G(\beta^*)$. Here, G maps β^* into $[0,1]$, see plot of G in Figure A.1 in the Proof of Proposition 1. Since $G(\beta)$ is bounded above by one, as $\lim_{\beta\to\infty} G(\beta) = 1$, only actions with $ak'(a) \leq 1$ are incentive compatible under an affine compensation scheme. The base wage, see (PC*), is then just set sufficiently high to ensure participation of the agent given her outside option $\underline{u}$ and cost of effort $k(a)$, where $\beta^*(1 - ak'(a)) > 0$ measures the agent's valuation of the variable pay component (in utils).

The contract parameters have the following intuitive properties, see plot in Figure 1. In this example, we set $k(a) = \frac{\psi^2}{2} a^2$ so that $\bar{a} = \frac{1}{\psi}$.[6]

Proposition 2 (Comparative Statics) *For $a \leq \bar{a}$, the optimal percentage bonus on the wage β^* is strictly increasing in the implemented action a while the base wage w^* is strictly decreasing in a. The compensation contract for the lower and upper bound is obtained in closed form.*

[6] The cost function determines the cutoff action $\bar{a}$, consistent with the results by Moroni and Swinkels (2014). By making ψ arbitrarily small, it is possible to make $\bar{a}$ arbitrarily large.

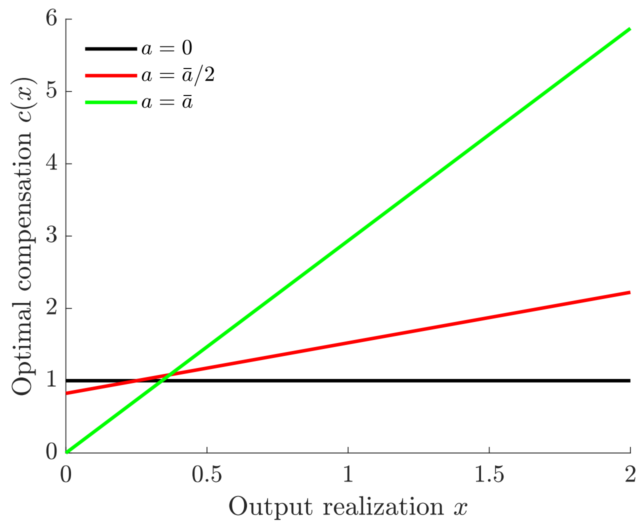

Figure 1. Optimal Affine Contracts. The graph plots the optimal compensation scheme $c(x)$ for 3 values of a between 0 and $\bar{a}$. For $a = \bar{a}$ (plotted in green), the compensation scheme is linear. The agent outside option is $\underline{u} = 0$ and the cost of effort satisfies $k(a) = 0.5a^2$ so that $\bar{a} = 1$.

- *For $a = 0$, the optimal contract consists only of a flat wage $c(x) = w^* = e^{\underline{u}}$.*
- *For $a = \bar{a}$, the optimal contract is a pure linear contract without a fixed wage*

$$c(x) = e^{\underline{u} + k(\bar{a}) + \gamma} \frac{x}{\bar{a}}, \tag{6}$$

where $\gamma \approx 0.58$ denotes the Euler-Mascheroni constant.

These comparative statics are intuitive. When no incentives need to be provided, $a = 0$, the optimal second-best contract corresponds to the first-best contract consisting only of a flat wage. The higher the action that the principal wants to implement, the higher the required incentives, which translates into higher β. However, even in the limit as β approaches infinity, incentives provided from a linear contract are finite as $\lim_{\beta \to \infty} G(\beta) = 1$. At $\bar{a}$, the optimal compensation contract is obtained in closed-form. The resulting second-best wage cost $W(\bar{a}) = W^{FB}(\bar{a}) e^{\gamma}$ are 58% higher than first-best compensation compensation costs of $W^{FB}(\bar{a}) = e^{\underline{u} + k(\bar{a})}$.

Proposition 1 implies that no solution to Problem 1 exists for $a > \bar{a}$ as no affine compensation contract can generate sufficient incentives. However, it does not rule out implementability with more convex compensation schemes. The following Lemma shows that supplementing fixed pay with call options (rather than a fixed equity share) is one way to generate sufficient convexity and implement effort levels above $\bar{a}$.

Lemma 2 (Implementability with Call Options) *For any exogenously given fixed wage of* $w < \exp[\underline{u} + k(a)]$*, actions* $a > \bar{a}$ *are implementable by additionally awarding* $\frac{w\beta}{a}$ *call options on output with a strike price equal to the* p*-quantile of the exponential distribution* $Q(p|a) := -a\ln[1-p]$*:*

$$c(x) = w + \frac{w\beta}{a}\max\left(x - Q(p|a), 0\right). \tag{7}$$

Given w*, the contract parameters* $p^* \in (0,1)$ *and* $\beta^* > 0$ *satisfy*

$$(1-p)\left(G(\beta) - (1-G(\beta))\,\beta\ln(1-p)\right) = ak'(a) \tag{IC**}$$
$$\ln(w) + (1-p)(1-G(\beta))\,\beta = \underline{u} + k(a) \tag{PC**}$$

The sole goal of Lemma 2 is show *implementability* of high effort levels. We defer the question of *optimality*, when and whether the principal would ever choose such a contract to the subsequent Proposition 3. Recall that the key constraint for implementability with affine contracts was incentive compatibility. Without restricting attention to affine incentives schemes, one way to satisfy (IC-FOC), $\mathbb{E}^a[L(x|a)\,u(c(x))] = k'(a)$, is to simply punish the agent sufficiently for negative likelihood ratios. Log utility implies that paying the agent close to the subsistence level of zero, allows the principal to punish the agent sufficiently. For the set of compensation contracts described in Lemma 2, the "punishment" occurs with probability p and corresponds to only paying a (sufficiently low) base wage $\underline{w} < \exp[\underline{u} + k(a)]$ as long as output falls below the p-quantile $Q(p,a)$. To see how (IC**) can now be satisfied, it is instructive to consider a first-order Taylor-series expansion about $p = 0$, which yields

$$(1-p)\,G(\beta) + p\,(1-G(\beta))\,\beta \approx ak'(a). \tag{8}$$

For $p = 0$, we obtain the baseline incentive compatibility condition (IC*), i.e., $G(\beta) = ak'(a)$. However, for any $p > 0$, one can now match the required effort incentives, $ak'(a)$, by setting β sufficiently high since $\lim_{\beta\to\infty}(1-G(\beta))\,\beta = \infty$.

While the class of contracts in Lemma 2 is just one way to implement effort levels $a > \bar{a}$, this class of contracts is of particular interest, as the following Proposition clarifies.

Proposition 3 *Suppose the principal solves Problem 1 for* $a > \bar{a}$ *and additionally faces a lower bound on compensation in the form of*

$$c(x) \geq \underline{w}, \tag{9}$$

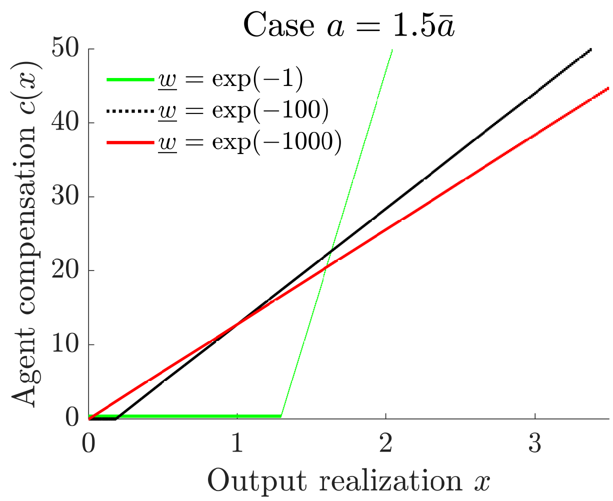

Figure 2. The graph plots optimal compensation contract for $a = 1.5\bar{a}$ for 3 different values of the lower bound on compensation $\underline{w}$.

where the minimum wage satisfies $\underline{w} < \exp\left[\underline{u} + k\left(a\right)\right]$*. Then, an optimal contract exists and is given by the contract in Proposition 2 setting* $w = \underline{w}$*. The punishment probability* p *(i.e., the option strike price) and the number of options,* $\frac{w\beta}{a}$*, are strictly increasing in* $\underline{w}$*.*

Proposition 3 offers two interpretations. First, from an applied perspective, bounds on compensation are not unrealistic, see e.g., Jewitt, Kadan and Swinkels (2008), and, hence this Proposition offers a potential rationale for the wide-spread use of call options. As a higher minimum wage limits the ability to "punish," the punishment region must optimally increase as must the number of options to provide sufficient incentives. Figure 2 illustrates these comparative statics by revealing that the kink moves to the right and the slope increases with the minimum wage.

Second, Proposition 3 allows us to shed light on the failure of existence in the original problem (absent a lower bound on compensation) by considering the limit as $\underline{w}$ becomes arbitrarily small. As is obvious, the principal is strictly better off as the constraint on the lower bound on pay is relaxed, i.e., as $\underline{w}$ approaches the agent's subsistence level of $\underline{c} = 0$.[7]

Lemma 3 (An almost linear contract!) *As the minimum wage approaches zero, the option strike price approaches zero, i.e.,* $\lim_{\underline{w}\to 0} p = 0$ *and*

$$\lim_{\underline{w}\to 0} \underline{w}\beta = \exp\left(\underline{u} + k\left(a\right) + \gamma + ak'\left(a\right) - 1\right). \tag{10}$$

[7] For generalized log utility $\ln\left(c - \underline{c}\right)$, the principal is constrained by the bound as long as $\underline{w} > \underline{c}$.

The infimum wage costs (absent a minimum wage) for any action $a > \bar{a}$ are given by

$$W_{\inf}(a) = e^{\underline{u}+k(a)+\gamma+ak'(a)-1}. \tag{11}$$

In the limit as $\underline{w} \to 0$ the contract is akin to a linear contract from the risk-neutral principal's perspective $\frac{x}{a}e^{\underline{u}+k(a)+\gamma+ak'(a)-1}$, see red line in Figure 2 for a "small" value of the minimum wage. Yet, if the principal were to write an exact linear contract, the agent would always choose $\bar{a}$ (and PC would be slack). The "missing" incentives of $ak'(a) - 1$ are created by the prospect of an infinitely severe (zero-pay) punishment realized with zero probability just so that (IC**) is satisfied, i.e., in the limit p goes to zero and β goes to infinity such that:

$$\lim_{\underline{w}\to 0} p\beta\left(1 - G(\beta)\right) = ak'(a) - 1. \tag{12}$$

The analysis so far considered the compensation design problem for a given action a, i.e., step 1 of the Grossman and Hart (1983) approach. We conclude by considering the action choice problem of the principal, step 2 of their approach. Giving that the principal obtains expected revenue a, the principal's action choice problem is

$$a^* = \arg\max_a a - W(a).$$

If one views non-existence as a "bug" rather than a feature (and one does not want to impose a lower bound), the closed-form solution for infimum wage costs makes it possible to derive sufficient conditions under which the principal would never want to implement actions $a > \bar{a}$.

Proposition 4 (Existence and affine contracts in equilibrium) *Suppose that the marginal effort cost function is convex, $k'''(a) \geq 0$, and the agent outside option is intermediate $\underline{u} \in [\check{\underline{u}}, \hat{\underline{u}}]$, then the equilibrium contract is affine as $a^* \leq \bar{a}$, see Proposition 1, and the principal's net profits are positive, $a^* - W(a^*) \geq 0$.*

The idea behind the proof is simple. Convexity of the marginal cost function, $k'''(a) \geq 0$, ensures that infimum wage costs, see (11), are strictly convex.[8] The outside option $\underline{u}$ raises both the marginal cost and the level thereof. The lower bound $\check{\underline{u}}$ ensures that the marginal compensation costs exceeds the marginal revenue, $W'_{\inf}(a) > 1$ for all $a > \bar{a}$. The upper bound $\hat{\underline{u}}$ ensures that the principal's net profits are still positive for $a = \bar{a}$, i.e., $\bar{a} > W(\bar{a})$. (See Proof of Proposition 4 for the threshold values). In combination,

[8] In general, Jewitt et al. (2008) show that conditions for the convexity of the principal's compensation cost function are much more complex.

Proposition 4 thus provides intuitive conditions for a "well-behaved" equilibrium with positive profits for the principal and optimal affine compensation contracts for the agent.

A Proofs

Proof of Lemma 1: First, we prove that an affine compensation scheme emerges from (1) for all values of $\underline{u}$ and cost functions $k(a)$ if and only if $\frac{1}{u'(c)}$ is affine in c and $L(x|\, a)$ is affine in x. Let $\eta(c) = 1/u'(c)$, then an affine contract requires that $\eta^{-1}(\lambda_{PC} + \lambda_{IC}\, L(x|\, a))$ is affine in x for any value of λ_{PC} and λ_{IC} (since these multipliers encode the outside option $\underline{u}$ and effort cost function $k(a)$). This requires that $\frac{1}{u'(c)}$ needs to be affine in c and $L(x|\, a)$ needs to be affine in x, i.e.,

$$\frac{1}{u'(c)} = k_1 + k_2 c \tag{A.1}$$

$$\frac{f_a(x|\, a)}{f(x|\, a)} = k_3 + k_4 x \tag{A.2}$$

where $f_a(x|\, a) := \frac{\partial f(x|a)}{\partial a}$. We impose the economic restrictions that $k_2 > 0$ (decreasing marginal utility) and $k_4 > 0$ (MLRP), see Milgrom (1981).

The solution to the ordinary differential equation (A.1) is given by

$$u(c) = C + \frac{\ln(k_1 + k_2 c)}{k_2}.$$

Since a monotonic transformation of a utility function preserves optimal choices, it is without loss of generality from an economic perspective to set $C = 0$, $k_2 = 1$ and $k_1 = -\underline{c}$, yielding generalized log utility, see Rubinstein (1977), i.e.,

$$u(c) = \ln(c - \underline{c}),$$

where $\underline{c}$ can be economically interpreted as the subsistence consumption level.

To obtain the solution the solution to the partial differential equation (PDE) in (A.2), it is useful to set $k_3 = \frac{\kappa'(a)}{\kappa(a)}$ and $k_4 = -\delta'(a)$ with $\delta'(a) < 0$ so that MLRP is satisfied for all values of a. The solution to the PDE can then be written as follows

$$f(x|\, a) = \kappa(a)\, z(x)\, e^{-\delta(a)x}. \tag{A.3}$$

To ensure that $f(x|\, a)$ is a valid density function, the product $\kappa(a)\, z(x)$ must be positively valued and normalized such that the integral over the support of x is equal to 1. One can now verify that any density function satisfying Condition (A.3) indeed produces an affine likelihood ratio, i.e., $\frac{f_a(x|a)}{f(x|a)} = \frac{\kappa'(a)}{\kappa(a)} - \delta'(a)\, x$. ■

Proof of Proposition 1: We first show that the first-order approach is valid for affine compensation contacts. To do so, it is sufficient to show that the compensation value of a linear contract from the agent's perspective, $V(\tilde{a}) := \mathbb{E}^{\tilde{a}}\left[\ln\left(w + w\beta\frac{x}{a}\right)\right]$, given a desired action a, is strictly concave in the agent action $\tilde{a}$. Integration and using the definition of $\operatorname{Ei}(y)$ yields

$$V(\tilde{a}) = \ln(w) - e^{\frac{a}{\tilde{a}}\frac{1}{\beta}} \operatorname{Ei}\left(-\frac{a}{\tilde{a}}\frac{1}{\beta}\right). \tag{A.4}$$

The goal is to show that $V''(\tilde{a}) < 0$. Let $z(\tilde{a}) := \tilde{a}\frac{\beta}{a} \geq 0$, then the second derivative of V can be written as follows

$$V''(\tilde{a}) = -\frac{\Theta(z(\tilde{a}))}{\tilde{a}^2}, \tag{A.5}$$

where the functions Θ and G satisfy

$$\Theta(y) := \left(\frac{1}{y} + 2\right) G(y) - 1, \tag{A.6}$$

$$G(y) := 1 + \frac{1}{y} e^{\frac{1}{y}} \operatorname{Ei}\left(-\frac{1}{y}\right). \tag{A.7}$$

Both functions Θ and G are strictly increasing in y and map $y \in \mathbb{R}^+$ into $[0,1]$, see plot in Figure A.1. Since $\Theta(z(\tilde{a})) \in [0,1]$ and $\tilde{a}^2 > 0$, we, thus obtain that $V''(\tilde{a}) < 0$. Due

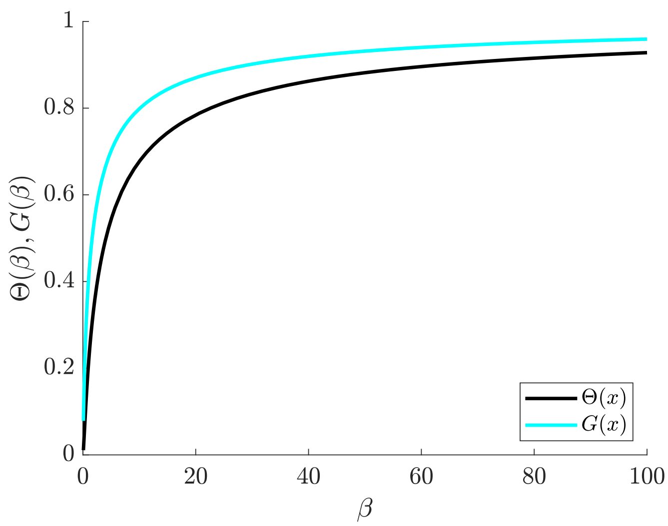

Figure A.1. This graph plots the functions $G(\beta) := 1 + \frac{1}{\beta} e^{\frac{1}{\beta}} \operatorname{Ei}\left(-\frac{1}{\beta}\right)$ and $\Theta(y) := \left(\frac{1}{y} + 2\right) G(y) - 1$.

to strict concavity of V, the agent's overall objective $V(\tilde{a}) - k(\tilde{a})$, is strictly concave for any convex cost function k. This means that (IC-FOC) is both necessary and sufficient for agent optimality.

Substituting the affine incentive scheme (3) into (IC-FOC) and integrating yields (IC*). Effort incentives are solely determined by β (and, hence, independent of the wage w), which is also evident from (A.4). Since $G(\beta) \in [0,1]$, see Figure A.1, a unique solution for β exists if and only if the right hand side of (IC*) is less than 1, i.e., $ak'(a) < 1$. Given that $ak'(a)$ is strictly increasing in a (by convexity of k) this condition is equivalent to $a < \bar{a}$. Hence, any action $a \in [0, \bar{a})$ is implementable with an affine compensation contract.

Given the solution for β^* from (IC*), we obtain the optimum base wage w^* from binding (PC), i.e., the agent's compensation value given the optimal action a, $V(a) = \ln(w) - e^{\frac{1}{\beta^*}} \operatorname{Ei}\left(-\frac{1}{\beta^*}\right)$, see (A.4), matches the agent's outside option $\underline{u} + k(a)$, so that

$$\ln(w^*) = \underline{u} + k(a) + e^{\frac{1}{\beta^*}} \operatorname{Ei}\left(-\frac{1}{\beta^*}\right). \tag{A.8}$$

Using the optimality condition (IC*) for β^*, we obtain (PC*). ■

Proof of Proposition 2: We first show the comparative statics of β. The optimality condition (IC*) implies that $G(\beta) = ak'(a)$. The implicit function theorem, thus, implies that

$$\frac{d\beta^*}{da} = \frac{k'(a) + ak''(a)}{G'(\beta)} > 0, \tag{A.9}$$

which is strictly positive because k is strictly increasing and convex and G is strictly increasing in β, i.e.,

$$G'(\beta) = \frac{\beta - G(\beta)(1+\beta)}{\beta^2} > 0. \tag{A.10}$$

We now turn to the comparative statics of the wage w. Taking the total derivative of (A.8) with respect to a yields:

$$\begin{aligned} \frac{d\ln w^*}{da} &= k'(a) - \frac{e^{\frac{1}{\beta^*(a)}} \operatorname{Ei}\left(-\frac{1}{\beta^*(a)}\right) + \beta^*(a)}{\beta^*(a)^2} \frac{d\beta^*}{da} \\ &= -\left(\frac{\Theta(\beta^*(a))}{G'(\beta^*(a))\,\beta^*(a)} k'(a) + \frac{G(\beta^*(a))}{G'(\beta^*(a))\,\beta^*(a)} ak''(a)\right) < 0. \end{aligned} \tag{A.11}$$

where the second line follows from substituting in $\frac{d\beta^*}{da}$ from (A.9) and using $G'(\beta)$ from (A.10). The derivative (A.11) is, thus, negative because Θ, G and G' are positive valued and the cost function k is strictly increasing and convex.

For $a = 0$, the compensation contract is simply the (flat) first-best compensation contract. Next consider the limit as a approaches $\bar{a}$, in which case $\lim_{a\to\bar{a}} \beta^*(a) = \infty$. Since $\lim_{\beta\to\infty} e^{\frac{1}{\beta}} \operatorname{Ei}\left(-\frac{1}{\beta}\right) = -\infty$, (A.8) implies that the log wage does so as well. The limiting compensation contract can, thus, be written expressed as

$$\lim_{a\to\bar{a}} c(x) = \lim_{a\to\bar{a}} w^* + w^*\beta^*\frac{x}{a} = \frac{x}{\bar{a}} \lim_{a\to\bar{a}} w^*\beta^*. \tag{A.12}$$

To determine $\lim_{a\to\bar{a}} w^*\beta^*$, it is useful to consider the limit of its logarithm:

$$\begin{aligned}\lim_{a\to\bar{a}} \ln\left(w^*\beta^*\right) &= \lim_{a\to\bar{a}} \ln w^* + \ln\beta^* \\ &= \underline{u} + k\left(\bar{a}\right) + \lim_{\beta\to\infty} e^{\frac{1}{\beta^*}}\,\mathrm{Ei}\left(-\frac{1}{\beta^*}\right) + \ln\beta^* \\ &= \underline{u} + k\left(\bar{a}\right) + \gamma, \end{aligned}\tag{A.13}$$

where the second line follows from (A.8) as well as $\lim_{a\to\bar{a}}\beta^*\left(a\right) = \infty$ and the third line follows from the fact that $\lim_{\beta\to\infty} e^{\frac{1}{\beta^*}}\,\mathrm{Ei}\left(-\frac{1}{\beta^*}\right) + \ln\beta^* = \gamma$, the Euler-Mascheroni constant. Since $\lim_{a\to\bar{a}} c\left(x\right) = \frac{x}{\bar{a}}\lim_{a\to\bar{a}} w^*\beta^*$ from (A.12) and using $\lim_{a\to\bar{a}}\ln\left(w^*\beta^*\right) = \underline{u} + k\left(\bar{a}\right) + \gamma$, we obtain the limit contract (6). ∎

Proof of Proposition 2: Given the piece-wise linear contract in (7), the payment to the agent is given by:

$$c\left(x\right) = \begin{cases} w & \text{if } x \le -a\ln[1-p] \\ w + \frac{w\beta}{a}\left(a\ln[1-p] + x\right) & \text{else} \end{cases}.\tag{A.14}$$

Since $\mathbb{P}^a\left[x \le -a\ln[1-p]\right] = p$, the agent, thus, receives w with probability p. Using (A.14) and integration yields

$$\mathbb{E}^a\left[L\left(x|\,a\right)u\left(c\left(x\right)\right)\right]a = \left(1-p\right)\left(G\left(\beta\right) - \left(1-G\left(\beta\right)\right)\beta\ln\left(1-p\right)\right)\tag{A.15}$$

$$\mathbb{E}^a\left[u\left(c\left(x\right)\right)\right] = u\left(w\right) + \left(1-p\right)\left(1-G\left(\beta\right)\right)\beta\tag{A.16}$$

(A.15) and (A.16) now directly imply (IC**) and (PC**). Because $\ln\left(1-p\right) < 0$ for any $p \in (0,1)$ and $\lim_{\beta\to\infty}\left(1-G\left(\beta\right)\right)\beta = \infty$, one can now set β sufficiently high so that (A.15) is satisfied. (PC**) now implies that

$$1 - p = \frac{\underline{u} + k\left(a\right) - \ln\left(w\right)}{\left(1-G\left(\beta\right)\right)\beta}.\tag{A.17}$$

Since $\underline{u} + k\left(a\right) > \ln\left(w\right)$, we obtain that $p < 1$. ∎

Proof of Lemma 3: The optimality of the contract described in Proposition 2 in the presence of a lower bound $\underline{w}$ follows from Definition 1 and Proposition 1 of Jewitt et al. (2008) as well as Proposition 2 of Hoffmann, Inderst and Opp (2020). That is, for $\lambda_{PC} + \lambda_{IC}\,L\left(x|\,a\right) < \underline{w}$, the lower bound binds, i.e., $c\left(x\right) = \underline{w}$. For $\lambda_{PC} + \lambda_{IC}\,L\left(x|\,a\right) \ge \underline{w}$, the constraint on the lower bound constraint is slack, and, hence the optimal payment is $c\left(x\right) = \lambda_{PC} + \lambda_{IC}\,L\left(x|\,a\right)$, where $L\left(x|\,a\right) = \frac{x-a}{a^2}$.

Using the compensation specification in (7), we can now perform comparative statics in $\underline{w}$. Substituting (A.17) into (IC**) implicitly characterizes the solution for β for any value of $\underline{w}$, i.e., $h\left(\beta,\underline{w}\right) = 0$, where

$$h\left(\beta,\underline{w}\right) = \frac{\frac{G(\beta)}{(1-G(\beta))\beta} + \ln\left(\frac{1-G(\beta)\beta}{\underline{u}+k(a)-\ln(\underline{w})}\right)}{\left(\underline{u} + k\left(a\right) - \ln\left(\underline{w}\right)\right)^{-1}} - ak'\left(a\right).\tag{A.18}$$

Letting $b = \beta \underline{w}$ and applying the implicit function theorem, we obtain

$$\frac{db}{d\underline{w}} = -\frac{\frac{\partial h}{\partial \underline{w}}}{\frac{\partial h}{\partial b}} = \frac{(1 - G(\beta))\,\beta\,(-\ln(1-p) - p) + pG(\beta)}{\beta G'(\beta)(1-p)} > 0 \tag{A.19}$$

Since $-\ln(1-p) > p$ for $p \in (0,1)$, $G(\beta) \in (0,1)$, $G'(\beta) > 0$, we obtain that $\frac{db}{d\underline{w}} > 0$.

Next we compute the comparative statics of p. Rearranging (A.17) and using $\beta = \frac{b}{\underline{w}}$, we obtain:

$$p = 1 - \frac{\underline{u} + k(a) - \ln(\underline{w})}{\left(1 - G\left(\frac{b}{\underline{w}}\right)\right)\frac{b}{\underline{w}}}. \tag{A.20}$$

Totally differentiating (A.20) with respect to $\underline{w}$ yields:

$$\frac{dp}{d\underline{w}} = \frac{\partial p}{\partial \underline{w}} + \frac{\partial p}{b}\frac{db}{d\underline{w}} = \zeta\left[\beta\left[1 - G(\beta)(1-p)\right] + G(\beta)(1-p)\frac{db}{d\underline{w}}\right] > 0, \tag{A.21}$$

where $\zeta = \frac{e^{-\frac{2}{\beta}}(1-G(\beta))}{\underline{w}\left[\mathrm{Ei}\left(-\frac{1}{\beta}\right)\right]^2} > 0$ and the term in brackets is positive since $0 < G(\beta) < 1$, $0 < p < 1$, and $\frac{db}{d\underline{w}} > 0$. ■

Proof of Lemma 3: The proof follows from considering the limit as the minimum wage approaches zero (from above):

$$\lim_{\underline{w}\downarrow 0} h\left(\frac{b}{\underline{w}}, \underline{w}\right) = 1 - \gamma - k(a) - \underline{u} + \ln[b] - ak'(a) \tag{A.22}$$

where γ denotes the Euler-Mascheroni constant. Setting $\lim_{\underline{w}\downarrow 0} h\left(\frac{b}{\underline{w}}, \underline{w}\right) = 0$ and solving for b, thus yields:

$$\lim_{\underline{w}\downarrow 0} b = \exp\left(\underline{u} + k(a) + \gamma + ak'(a) - 1\right).$$

We can now also solve for the limit Lagrange-multipliers as $\underline{w}$ approaches zero:

$$\lambda_{PC} + \lambda_{IC}\frac{x-a}{a^2} = \frac{b}{a}x \tag{A.23}$$

Matching coefficients implies that $\lim_{\underline{w}\downarrow 0}\lambda_{IC} = ab$ and $\lim_{\underline{w}\downarrow 0}\lambda_{PC} = b$. ■

Proof: Given infimum compensation costs of $W_{\inf}(a) = e^{\underline{u}+k(a)+\gamma+ak'(a)-1} > 0$, the first and second derivative are

$$W'_{\inf}(a) = W_{\inf}(a)\left(2k'(a) + ak''(a)\right), \tag{A.24}$$

$$W''_{\inf}(a) = W_{\inf}(a)\left(4k'(a)^2 + 3k''(a) + 4ak'(a)k''(a) + a^2k''(a)^2 + ak'''(a)\right) \tag{A.25}$$

(A.25) implies that convexity of the marginal cost function, $k'''(a) > 0$, ensures convexity of the infimum compensation cost function $W''_{\inf}(a) > 0$.

First, to ensure that the principal never wants to implement actions $a > \bar{a}$, it is sufficient that marginal compensation costs exceed the marginal revenue $W'_{\inf}(a) > 1$ for all $a > \bar{a}$. Given strict convexity of $W_{\inf}(a)$, this condition requires that $W'_{\inf}(\bar{a}) \geq 1$, i.e.,

$$e^{\underline{u}+k(\bar{a})+\gamma}\left(2k'(\bar{a})+ak''(\bar{a})\right) \geq 1. \tag{A.26}$$

Second, the condition $W_{\inf}(\bar{a}) < \bar{a}$ ensures that the principal's profits are positive for some $a \leq \bar{a}$, i.e.,

$$e^{\underline{u}+k(\bar{a})+\gamma} \leq \bar{a}. \tag{A.27}$$

Jointly, conditions (A.26) and (A.27), thus require that:

$$\frac{1}{2k'(\bar{a})+\bar{a}k''(\bar{a})} \leq e^{\underline{u}+k(\bar{a})+\gamma} \leq \bar{a}. \tag{A.28}$$

We first prove that $\bar{a} > \frac{1}{2k'(\bar{a})+\bar{a}k''(\bar{a})}$ or equivalently $\bar{a}\left(2k'(\bar{a})+\bar{a}k''(\bar{a})\right) > 1$. Since $\bar{a}k'(\bar{a}) = 1$, this condition is always satisfied as $1+\bar{a}^2k''(\bar{a}) > 0$ by convexity of k. Since $\bar{a} > \frac{1}{2k'(\bar{a})+\bar{a}k''(\bar{a})}$, it is thus possible to satisfy Condition (A.28) by choosing $\underline{u}$ appropriately. The lower and upper bound for the outside option are given by

$$\check{\underline{u}} = -\ln\left(2k'(\bar{a})+\bar{a}k''(\bar{a})\right) - k(\bar{a}) - \gamma, \tag{A.29}$$

$$\hat{\underline{u}} = \ln(\bar{a}) - k(\bar{a}) - \gamma. \tag{A.30}$$

■

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