Online-Appendix to "Only time will tell"

Appendix B Continuous action set

B.1 Optimal compensation design

In this Appendix we extend our setup to a continuous action set, $a \in \mathscr{A} = [0, \bar{a}]$. For simplicity, we restrict attention to the case of a risk-neutral agent and no upper bounds on payments. Formally, we now consider a family of filtered probability spaces $\left\{(\Omega, \mathcal{F}^X, (\mathcal{F}_t^X)_{0\le t\le \bar{T}}, \mathbb{P}^a); a \in [0, \bar{a}]\right\}$, and, to avoid degeneracies assume that $\mathbb{P}^{a_1}$ is equivalent to $\mathbb{P}^{a_2}$ for all $a_1, a_2 \in [0, \bar{a}]$.[32] The associated cost function $k(a)$ satisfies the usual conditions, i.e., it is strictly increasing and strictly convex with $k(0) = k'(0) = 0$ as well as $k'(\bar{a}) = \infty$. To mirror the structure of the analysis in the main text, we will, first, focus on optimal compensation design, i.e., characterize cost-minimizing contracts to implement a given action a (the first problem in Grossman and Hart (1983)). We return to the problem of the optimal action choice by the principal at the end of this Appendix.

As is common also in static moral hazard problems with continuous actions (see e.g., Holmstrom (1979) and Shavell (1979)) we assume that the first-order approach is valid and provide a sufficient condition in Theorem B.1 below. Hence, for each a, we replace (IC) by the following first-order condition

$$\frac{\partial}{\partial a}\mathbb{E}^a\left[\int_0^{\bar{T}} e^{-r_A t} db_t\right] = k'(a). \tag{IC-FOC}$$

As now local incentives matter according to (IC-FOC), the appropriate measure of agent performance, analogous to the likelihood ratio in the binary action case, is the *score* function which measures the (local) sensitivity of the likelihood function with respect to the action. Formally, denoting by $\mathbb{P}_t^a$ the restriction of $\mathbb{P}^a$ to $\mathcal{F}_t^X$, we define for each $a > 0$ the likelihood function $\mathcal{L}_t(a|\omega) := \frac{d\mathbb{P}_t^a}{d\mathbb{P}_t^0}(\omega)$ which exists from the Radon-Nikodym Theorem. To illustrate the close connection to the binary-action case we then denote the score by

$$L_t(a) := \frac{\partial}{\partial a}\log \mathcal{L}_t(a|\omega).$$

Here, we impose standard Cramér-Rao regularity conditions used in statistical inference (cf., e.g., Casella and Berger (2002)) by stipulating, in particular, that the score $L_t(a)$ exists and is bounded for any (t, ω). Then, analogous to the binary action case, the upper

[32] Two measures $\mathbb{P}^{a_1}$ and $\mathbb{P}^{a_2}$ are equivalent if $\mathbb{P}^{a_1} \ll \mathbb{P}^{a_2}$ and $\mathbb{P}^{a_2} \ll \mathbb{P}^{a_1}$.

support of the date-t score distribution, $\bar{L}_t(a)$,[33] will be the relevant measure of informativeness quantifying the information benefit of deferral in the setting with a continuous action set, where we again assume that $\bar{L}_t(a)$ exists and has positive probability mass (cf., Condition 1). Now, since the score is a martingale, $\bar{L}_t(a)$ is an increasing function of time (cf., Observation 1).

It is then immediate that our preceding characterization readily extends to the continuous action case with $\bar{L}_t$ replaced by $\bar{L}_t(a)$. For completeness, the following Theorem provides a characterization of the optimal contract including a sufficient condition for the validity of the first-order approach.[34]

Theorem B.1 *Assume that $\mathbb{P}^{\tilde{a}}\left(L_t(a) = \bar{L}_t(a)\right) > 0$ is strictly concave in $\tilde{a}$ for all t and that $\bar{L}_t(a) \leq \frac{k'(a)}{R+k(a)}$.*
1) If $R \leq \bar{R}(a) = \frac{k'(a)}{\bar{L}_{\hat{T}(a)}(a)} - k(a)$, (PC) is slack, and it is optimal to stipulate a performance bonus at a single date $\hat{T}(a) = \min\{\arg\max_t e^{-\Delta r t}\bar{L}_t(a)\}$.
2) Otherwise, (PC) binds, and payments are optimally made at maximally two payout dates which are characterized as follows: If there exists a date $T_1(a)$ such that $\bar{L}_{T_1(a)}(a) = \frac{k'(a)}{R+k(a)}$ and $C\left(\frac{k'(a)}{R+k(a)}\middle| a\right) = e^{\Delta r T_1(a)}$,[35] this is the unique payout date. Else, the contract requires a short-term payout date $T_S(a)$ and a long-term date $T_L(a)$ such that $\hat{T}(a) \geq T_L(a) > T_S(a) \geq 0$.

Proof of Theorem B.1. The arguments are identical to the proof of Theorem 1 and Lemma 1 given that the first-order approach is valid, which is ensured by the assumption that $\mathbb{P}^{\tilde{a}}\left(L_t(a) = \bar{L}_t(a)\right) > 0$ is strictly concave in $\tilde{a}$ for all t.[36] To see this, note that, given a contract as characterized in Theorem B.1, the agent's problem $\max_{\tilde{a}}\left\{\mathbb{E}^{\tilde{a}}\left[\int_0^{\bar{T}} e^{-r_A t} db_t\right] - k(a)\right\}$ is strictly concave. Hence, the first-order condition in (IC-FOC) is both necessary and sufficient for incentive compatibility. **Q.E.D.**

[33] The score is no longer bounded above by one, but this is irrelevant for the further analysis.
[34] The condition is reminiscent of the convexity of the distribution function condition (CDFC) in static models (cf. e.g., Rogerson (1985)).
[35] In analogy to the binary action case the cost of informativeness $C(\cdot|\, a)$ is defined as the lower hull of the set $\cup_{t=0}^{\bar{T}}\left(\bar{L}_t(a), e^{\Delta r t}\right)$.
[36] In fact, this sufficient condition is stronger than needed. It would suffice to assume strict concavity in a at the relevant optimal payout dates characterized in Theorem B.1 ($\hat{T}(a)$, $T_1(a)$ or $T_S(a)$ and $T_L(a)$).

B.2 Optimal action choice

So far, the analysis has focused on the principal's costs to induce a given action[37]

$$W(a) = \begin{cases} \frac{k'(a)}{\bar{L}_{\hat{T}(a)}(a)} e^{\Delta r \hat{T}(a)} & R \leq \bar{R}(a) \\ (R + k(a))\, C\left(\frac{k'(a)}{R+k(a)} \middle| a \right) & R > \bar{R}(a). \end{cases}$$

We now shortly discuss the principal's preferences over actions and the resulting equilibrium action choice, the second problem in Grossman and Hart (1983). We capture the benefits of an action a to the principal by a strictly increasing and concave bounded function $\pi(a)$. Here, $\pi(a)$ could correspond to the *present value* of the (gross) profit streams under action a. Given any (gross) profits $\pi(a)$ and compensation costs $W(a)$ the equilibrium action then solves

$$a^* = \arg\max_{a \in \mathscr{A}} \pi(a) - W(a),$$

and, given a solution a^*, the chosen payout times are characterized by Theorem B.1.[38]

References

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[37] If $\bar{L}_0(a) > \frac{k'(a)}{R+k(a)}$ the incentive constraint is slack and all payments are optimally made at time 0 resulting in wage costs of $R + k(a)$.

[38] If $\bar{L}_0(a^*) > \frac{k'(a^*)}{R+k(a^*)}$ the optimal payout date is at $t = 0$.